

The University of Chicago

DISCONTINUOUS SOLUTIONS FOR
THE PROBLEM OF BOLZA IN
PARAMETRIC FORM

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1937

By
MALCOLM FINLAY SMILEY

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CONTRIBUTIONS TO THE CALCULUS OF VARIATIONS, 1933-37
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IN PARAMETRIC FORM

Introduction. In the customary treatment of a simple integral problem in the calculus of variations the assumption is made for certain important parts of the theory that the minimizing arc has no corners. For the simplest problem in n -space and in parametric form Graves (III)¹ showed that a theory quite similar to the usual one could be given for minimizing arcs with corners, the so-called "discontinuous solutions." Some of his results were generalized by Hefner (IV) to the problem of Lagrange with fixed end-points and in parametric form. More recently Reid (IX) showed that discontinuous solutions of the general problem of Mayer in non-parametric form could be treated by suitably modifying the usual procedure. It is the purpose of this paper to discuss discontinuous solutions of the problem of Bolza in parametric form. An attempt has been made to follow the development of the theory of the non-parametric problem of Bolza as presented by Bliss (II).

The paper has been arranged so as to place emphasis on the principal theorem, a sufficiency theorem proved in Sections 9-12. Sections 1 and 2 are preliminary. Necessary conditions which follow immediately from the non-parametric theory are stated in Section 3. Section 4 is devoted to the construction of a family of extremaloids embedding a given non-singular extremaloid.

¹The Roman numerals refer to the bibliography at the end of the paper.

The consideration of the second variation and formulation of a Jacobi, or fourth, necessary condition occupy Sections 5 and 6. In Sections 7 and 8 certain interesting equivalences between the strengthened forms of the Jacobi condition are established. These results permit an economical statement of the sufficiency theorem, the proof of which concludes the paper.

1. Statement of the problem. The problem of Bolza in parametric form is that of finding in a class of admissible arcs

$$(1:1) \quad y_i = y_i(t) \quad (i = 1, \dots, n; t_1 \leq t \leq t_2),$$

which satisfy equations of the form

$$(1:2) \quad \varphi_\beta(y, y') = 0 \quad (\beta = 1, \dots, m < n - 1),$$

$$(1:3) \quad \psi_\mu[y(t_1), y(t_2)] = 0 \quad (\mu = 1, \dots, p \leq 2n),$$

one which gives the functional

$$J = g[y(t_1), y(t_2)] + \int_{t_1}^{t_2} f(y, y') dt$$

a minimum value. We shall also consider here a problem in which the equations (1:3) are replaced by

$$(1:4) \quad \psi_\rho[y(t_1)] = 0 \quad (\rho = 1, \dots, \rho_0),$$

$$(1:5) \quad \psi_\sigma[y(t_2)] = 0 \quad (\sigma = 1 + \rho_0, \dots, \sigma_0 + \rho_0 \leq 2n),$$

and for which the function g has the form

$$g_1[y(t_1)] - g_2[y(t_2)].$$

The second formulation will be referred to as the problem with separated end-conditions. (Cf. VI, p. 391.)

Throughout this paper, unless the contrary is stated explicitly, the following subscripts will have the indicated ranges

$$\begin{aligned} i, j &= 1, \dots, n; & k, l &= 1, \dots, n-1; \\ \mu &= 1, \dots, p; & \beta &= 1, \dots, m; & s &= 1, 2. \end{aligned}$$

We shall adopt the summation convention of tensor analysis modified as usual for use in the calculus of variations.

2. Analytic basis. We shall state here hypotheses sufficient to carry through our analysis of the problem stated in Section 1. We shall assume that there is an open region $\mathcal{R}_1$ of points $(y, r) = (y_1, r_1)$ with $r_1 \neq 0_1$ in which the following conditions are satisfied:

- (a) If the set $(y, r) = (y_1, r_1)$ is in $\mathcal{R}_1$ and k is positive then the set $(y, kr) = (y_1, kr_1)$ is in $\mathcal{R}_1$.
- (b) The functions $f(y, r)$ and $\varphi_\beta(y, r)$ are of class C''' and positively homogeneous of degree one in r in $\mathcal{R}_1$.
- (c) The matrix $\|\varphi_{\beta r_1}(y, r)\|$ has rank m in $\mathcal{R}_1$.

We assume further that there is an open region $\mathcal{R}_2$ of $2n$ -dimensional space in which the functions $g[y_{11}, y_{12}]$ and $\gamma_\mu[y_{11}, y_{12}]$ are of class C''' and the matrix $\|\gamma_{\mu y_{11}} \quad \gamma_{\mu y_{12}}\|$ has rank p .

An arc defined by equations of the form (1:1) which is of class D' on its interval of definition, which defines elements (y, y') in $\mathcal{R}_1$, which satisfies the equations (1:2), and whose end-elements are in $\mathcal{R}_2$ will be called an admissible arc. We shall allow transformations of the parameter of the form

$$t = T(\bar{t}) \qquad (\bar{t}_1 \leq t \leq \bar{t}_2),$$

where T is continuous and piecewise of class C'' , and T' is positive. It is then seen, in view of the assumption (b), that the

equations (1:2), (1:3), and admissibility are properties of the arc independent of allowable transformation of the parameter. It is also clear that the value of the functional J is determined by the arc.

3. First necessary conditions. We shall as usual center our attention on a particular admissible arc given by equations of the form (1:1). Let $F(y, r, \lambda) = \lambda_0 f(y, r) + \lambda_\beta \varphi_\beta(y, r)$. An arc (1:1) is said to satisfy condition I with a set of multipliers $\lambda_0 = \text{constant}$, $\lambda_\beta(t)$ ($t_1 \leq t \leq t_2$) in case the set $[y(t), y'(t), \lambda_\beta(t)]$ satisfies with constants c_1 the Du Bois Reymond equations

$$(3:1) \quad F_{r_1} = \int_{t_1}^{t_2} F_{y_1} dt + c_1, \quad \varphi_\beta = 0,$$

if further there are constants e_μ such that the transversality condition

$$(3:2) \quad F_{r_1} dy_1|_1^2 + \lambda_0 dg + e_\mu d\gamma_\mu = 0$$

holds for all dy_{11} , dy_{12} , and if finally the multipliers $\lambda_\beta(t)$ are continuous between corners of this arc, are of class D on $t_1 t_2$, and the functions λ_0 , $\lambda_\beta(t)$ do not vanish simultaneously. If the arguments of the functions F , φ_β , and their partial derivatives are not stated specifically it is to be understood that they are the set $[y_1(t), y_1'(t), \lambda_\beta(t)]$. From the non-parametric theory it follows at once that a minimizing arc must satisfy the condition I (II, p. 26).

For an arc satisfying condition I the Euler-Lagrange equations,

$$(d/dt)F_{r_1} = F_{y_1}, \quad \varphi_\beta = 0,$$

hold between corners on the arc defined by $t = x_\theta$ ($\theta = 1, \dots, r$) at which the Weierstrass-Erdmann corner conditions $F_{r_1}(x_\theta^-) = F_{r_1}(x_\theta^+)$ must hold.¹ Such an arc and set of multipliers will be called an extremaloid. At each point not a corner of the arc where the determinant

$$R = \begin{vmatrix} F_{r_1 r_j} & \varphi_{\beta r_1} & y_1' \\ \varphi_{\gamma r_j} & 0_{\gamma \beta} & 0_\gamma \\ y_j' & 0_\beta & 0 \end{vmatrix}$$

is different from zero the functions $y_1(t)$, $\lambda_\beta(t)$ may be chosen of class C''' , C'' , respectively (IV, p. 7).

The order of abnormality of an arc is the maximum number of linearly independent sets of multipliers $\lambda_0 = 0$, $\lambda_\beta(t)$ with which the arc satisfies the condition I. When the abnormality is zero the arc is said to be normal and in this case we can choose the multiplier λ_0 as unity, the remaining multipliers then being unique. The arc is said to be normal on a subinterval in case it is normal relative to arcs joining its ends on the subinterval.

Let $\mathcal{E}(y, r, \lambda; \bar{r}) = F(y, \bar{r}, \lambda) - \bar{r}_1 F_{r_1}(y, r, \lambda)$. An extremaloid $E: y_1(t)$, $\lambda_0 = 1$, $\lambda_\beta(t)$ ($t_1 \leq t \leq t_2$) will be said to satisfy the condition II (II') in case the inequality

$$\mathcal{E}[y(t), y'(t), \lambda(t); \bar{r}] \geq 0 \quad (> 0)$$

holds for all $(y, \bar{r}) \neq (y, ky'^-)$, $(y, \bar{r}) \neq (y, ky'^+)$ ($k > 0$).

The extremaloid E satisfies the condition III (III') if along E the inequality

¹Throughout this paper the superscripts +, - attached to a variable will indicate the right- and left-hand limits, respectively, of the function involved.

$$F_{r_1 r_j} \pi_1 \pi_j \geq 0 \quad (> 0)$$

holds for all sets $(\pi) \neq (ky')$ satisfying $\Phi_{\theta r_1} \pi_1 = 0$. It follows as before from the non-parametric theory that a normal minimizing arc satisfying the condition I with multipliers $\lambda_0 = 1$, $\lambda_{\theta}(t)$ must satisfy the conditions II and III.

We shall conclude this section with the statement of a relation between the function $\Omega_0(t)$ defined by the equation

$$\Omega_0(t) = F_{y_1}(t^+) y_1'(t^-) - F_{y_1}(t^-) y_1'(t^+)$$

and the $\mathcal{E}$ -function. Hefner showed that for an extremaloid along which we have $\mathcal{E} \geq 0$, we also have $\Omega_0 \leq 0$ at the corners of the arc (IV, pp. 13-14). It is then clear that at the corners of a normal minimizing arc satisfying the condition I with multipliers $\lambda_0 = 1$, $\lambda_{\theta}(t)$ we must have $\Omega_0 \leq 0$.

4. An embedding theorem. An extremaloid $E: y_1(t)$, λ_0 , $\lambda_{\theta}(t)$ will be said to be non-singular in case the determinant R is different from zero along it and the function Ω_0 is different from zero at its corners. We shall show in this section how to embed a non-singular extremaloid E in a family of extremaloids. We first choose the parameter so that $y_1' y_1' = 1$ and introduce canonical variables (y, z) by means of the equations

$$\begin{aligned} z_1 &= F_{r_1}(y, r, \bar{\lambda}) + \mu r_1, \\ (4:1) \quad 0 &= \Phi_{\theta}(y, r), \\ 1 &= r_1 r_1. \end{aligned}$$

These equations have the initial solution $(y, r, \bar{\lambda}, z, \mu) = (y, y', \lambda, F_{r_1}(y, y', \lambda), 0)$ at which their functional determinant with respect to $r, \bar{\lambda}, \mu$ is R . By the usual implicit function theorems they may be solved in a neighborhood S of the

values (y, z) belonging to the extremal subarc. The solutions

$$(4:2) \quad r_1 = P_1(y, z), \quad \bar{\lambda}_\beta = \Lambda_\beta(y, z), \quad \mu = M(y, z)$$

reduce to $y', \lambda, 0$ at each point (y, z) belonging to E , and are of class C'' in S . In the neighborhood S let us define a Hamiltonian function $H(y, z)$ by the equation

$$H(y, z) = z_1 P_1 - F(y, P, \Lambda).$$

By direct calculation we find that

$$H_{y_1} = -F_{y_1}, \quad H_{z_1} = P_1.$$

From these equations it is evident that in S the equations

$$(4:3) \quad y_1' = H_{z_1}, \quad z_1' = -H_{y_1}$$

are equivalent to the equations

$$(4:4) \quad y_1' = P_1, \quad z_1' = F_{y_1}.$$

Using the following well-known identity in $y_1(t), \lambda_\beta(t)$

$$y_1' [(d/dt)F_{r_1} - F_{y_1}] = \mathcal{P}_\beta \lambda'_\beta$$

(VI, p. 397) it is easily seen that the function $M(y, z)$ is a constant along a solution of the equations (4:3). By differentiating the equations (4:1) we find that $M_{z_1} = P_1$. Since not all of the functions P_1 vanish at any point of the subarc, there is a subinterval of the values of t defining a portion of the extremal subarc and a neighborhood of values (y, z) belonging to this portion of the extremal subarc in which one of them, say P_n , remains different from zero. By implicit function theory we may solve the equation $M(y, z) = 0$ for z_n in the neighborhood just described. It follows that for arbitrary choice of the initial

values $y_{10}, \dots, y_{n0}, z_{10}, \dots, z_{n-10}$ in this neighborhood there is an extremal arc taking on these values. Since all of these arcs intersect the hyperplane $y_n = y_{n0}$, we lose none of them by fixing this value. Application of the usual embedding theorem for equations of the type (4:3) gives us a $(2n-2)$ -parameter family of extremals in canonical form

$$y_1 = y_1(t, a, b), \quad z_1 = z_1(t, a, b)$$

embedding the first extremal subarc of E . We denote by a_{k0}, b_{k0} the values of the parameters defining E .

Let us now consider the corner equations

$$\begin{aligned} F_{r_1}[y(X, a, b), p, \bar{\lambda}] - F_{r_1}[y(X, a, b), y'(X, a, b), \Lambda(X, a, b)] &= 0, \\ (4:5) \quad \varphi_\beta[y(X, a, b), p] &= 0, \\ p_1 p_1 &= 1. \end{aligned}$$

These equations have the initial solution,

$$\begin{aligned} X = x_1, \quad a_k = a_{k0}, \quad b_k = b_{k0}, \quad p_1 = y_1'(x_1^+), \quad \bar{\lambda}_\beta = \lambda_\beta(x_1^+) \\ (k = 1, \dots, n-1; i = 1, \dots, n; \beta = 1, \dots, m) \end{aligned}$$

at which the functional determinant with respect to $p, \bar{\lambda}, X$ is $2R\Omega_0(x_1)$ (IV, p. 11). Since E is non-singular the usual implicit function theory applies and there is a neighborhood T of this initial solution in (a, b, x, p, λ) -space and functions

$$X = t(a, b), \quad p_1 = p_1(a, b), \quad \bar{\lambda}_\beta = \lambda_\beta(a, b)$$

which are the unique solutions of the equations (4:5) in the neighborhood T . Let us employ the values

$$\begin{aligned} y_{10} &= y_1[t(a, b), a, b], \\ z_{10} &= F_{r_1}[y_0, p(a, b), \lambda(a, b)] \end{aligned}$$

as initial values of the canonical equations determined by the second extremal subarc of E . These initial values determine an extremal as is evident from the uniqueness of the solutions of the equations (4:1).

Proceeding in this way we see that we may embed a non-singular extremaloid for fixed value of the multiplier λ_0 in a $(2n - 2)$ -parameter family of extremaloids in canonical form,

$$(4:6) \quad y_1 = y_1(t, a, b), \quad z_1 = z_1(t, a, b).$$

Using a device employed by Graves (III, p. 7) we can, by suitable transformation of the parameter on the subfamilies, obtain a family for which the corners occur on the various arcs for the same values of the parameter. The corner manifolds of such a family are then given by equations of the form

$$y_1 = y_1(x_\theta, a, b) \quad (\theta = 1, \dots, r)$$

and it is a consequence of the preceding argument that these manifolds are of class C^1 in a neighborhood of the values (a_0, b_0) . When the parameter is chosen in this way the partial derivatives of the functions (4:6) with respect to the constants a_k, b_l are continuous on $t_1 t_2$.

We summarize the results of this section in the following

THEOREM 4.1. AN EMBEDDING THEOREM. Every non-singular extremaloid E for fixed value of the multiplier λ_0 is embedded for values

$$t_1 \leq t \leq t_2, \quad a_k = a_{k0}, \quad b_l = b_{l0} \quad (k, l = 1, \dots, n - 1),$$

in a $(2n - 2)$ -parameter family of extremaloids defined by functions of the form

$$(4:7) \quad y_1 = y_1(t, a, b), \quad z_1 = z_1(t, a, b).$$

The functions y_1, y_{1t}, z_1, z_{1t} are of class C^n between corner manifolds,

$$y_1 = y_1(x_\theta, a, b) \quad (\theta = 1, \dots, r)$$

in a neighborhood of the values (t, a, b) defining E . The partial derivatives $y_{1a_k}, z_{1a_k}, y_{1b_k}, z_{1b_k}$ are continuous on the interval $t_1 t_2$.

5. The second variation and the accessory minimum problem

A set of functions $\eta_1(t)$ is called a set of admissible variations in case they have the continuity properties of an admissible arc and satisfy the equations

$$\Phi_p(\eta, \eta') = \Phi_{p y_1} \eta_1 + \Phi_{p r_1} \eta_1' = 0.$$

In terms of our original problem we may define a quadratic form

$$2\omega(\eta, \pi) = F_{y_1 y_j} \eta_i \eta_j + 2F_{y_1 r_1} \eta_i \pi_i + F_{r_1 r_j} \pi_i \pi_j,$$

and a functional known as the second variation

$$J_2(\eta) = 2\delta [\eta(t_1), \eta(t_2)] + \int_{t_1}^{t_2} 2\omega(\eta, \eta') dt,$$

where 2δ is a quadratic form whose coefficients are the second derivatives of the function $g + e_\mu \gamma_\mu$ evaluated on E . It follows from the theory of the non-parametric problem that for a normal minimizing arc the second variation must be non-negative in the class of admissible variations which have second derivatives between a finite number of corners and which satisfy the equations

$$(5:1) \quad \mathcal{F}_\mu(\eta) = \gamma_{\mu y_{1s}} \eta_1(t_s) = 0 \quad (s = 1, 2)$$

(II, p. 72). By the notation IV_* we shall mean that the second variation is non-negative in this class and vanishes only for tangential variations. The term tangential variation denotes a set of admissible variations of the form $\rho y_1'$ with ρ continuous on t_1, t_2 . We are led, by the methods originated by Bliss, to the consideration of an accessory minimum problem, that is, to the problem of minimizing $J_2(\eta)$ in the class described. The problem as stated is not precisely in the Bolza form, but we may state an equivalent problem which is in this form exactly as does Bliss (II, pp. 79-80). Although the coefficients of ω and Φ_ρ are discontinuous, the necessary conditions may still be deduced because of the linearity of the equations involved. The extremals for this problem satisfy between corners the equations

$$(5:2) \quad (d/dt)\Omega_{\pi_1} = \Omega_{\eta_1}, \quad \Phi_\rho = 0,$$

and at corners the corner conditions,

$$\Omega_{\pi_1}(t^+) = \Omega_{\pi_1}(t^-),$$

where $\Omega(\eta, \pi, \mu) = \omega(\eta, \pi) + \mu_\rho \Phi_\rho(\eta, \pi)$. The arguments of Ω, Φ_ρ , and their partial derivatives will be the set $[\eta_1(t), \eta_1'(t), \mu_\rho(t)]$ unless the contrary is stated explicitly. The transversality condition (3:2) becomes

$$(5:3) \quad \Omega_{\pi_1} d\eta_1|_1^2 + d\gamma + \epsilon_\mu d\pi_\mu = 0$$

for all $d\eta_{11}, d\eta_{12}$.

Canonical variables may be defined by the equations

$$(5:4) \quad \begin{aligned} \zeta_1 &= \Omega_{\pi_1}(\eta, \pi, \mu) + \nu y_1', \\ 0 &= \Phi_\rho(\eta, \pi), \\ 0 &= y_1' \pi_1, \end{aligned}$$

whose second members are linear in η_1 , π_1 , μ_ρ , ν and have the determinant of coefficients of π_1 , μ_ρ , ν equal to R. If we assume that E is non-singular this determinant is not zero and the equations (5:4) have solutions

$$(5:5) \quad \pi_1 = \Pi_1(\eta, \xi), \quad \mu_\rho = M_\rho(\eta, \xi), \quad \nu = N(\eta, \xi),$$

linear and homogeneous in the variables η_1 , ξ_1 . A Hamiltonian function for the second variation may be defined by the equation

$$(5:6) \quad \mathcal{H}(\eta, \xi) = \xi_1 \Pi_1 - \Omega(\eta, \Pi, M).$$

We see that $\mathcal{H}$ is a quadratic form and that we have

$$\mathcal{H}_{\eta_1} = -\Omega_{\eta_1}, \quad \mathcal{H}_{\xi_1} = \Pi_1.$$

It follows immediately that the canonical accessory equations

$$(5:7) \quad \eta_1' = \mathcal{H}_{\xi_1}, \quad \xi_1' = -\mathcal{H}_{\eta_1}$$

are equivalent by the definitions (5:5) to the equations

$$(5:8) \quad (d/dt)(\Omega_{\pi_1} + \nu y_1') = \Omega_{\eta_1}, \quad \Phi_\rho = 0, \quad y_1' \eta_1' = 0.$$

By an accessory extremaloid we shall mean a set of functions $\eta_1(t)$, $\xi_1(t)$ which satisfy the conditions

(a) η_1 , ξ_1 are continuous on $t_1 t_2$ and of class C' on

$x_\theta < t < x_{\theta+1}$ ($\theta = 0, \dots, r$), where $x_0 = t_1$, $x_{r+1} = t_2$.

(b) There are multipliers $\mu_\rho(t)$ such that $\xi_1 = \Omega_{\pi_1}$.

We note that for the choice of the parameter on the arc explained in Section 4 the partial derivatives of the family (4:7) with respect to the constants of integration are accessory extremaloids. It is also to be noted that for an accessory extremaloid η_1 , ξ_1 we have $\xi_1 y_1' - \eta_1 F_{y_1} \equiv 0$.

Closely related to the accessory extremaloids are the pseudo-accessory extremaloids which consist of constants ξ_θ ($\theta = 1, \dots, r$) and functions $\eta_1(t)$, $z_1(t)$ satisfying the conditions

(a'') η_1 , z_1 are of class C^1 and satisfy equations (5:7)

on $x_\theta < t < x_{\theta+1}$ ($\theta = 0, \dots, r$).

$$(b') \quad \xi_{\theta y_1'} + \eta_1 \Big|_{t=x_\theta^+}^{t=x_\theta^-} = 0$$

($\theta = 1, \dots, r$),

$$\xi_{\theta F_{y_1}} + z_1 \Big|_{t=x_\theta^+}^{t=x_\theta^-} = 0$$

$$(c') \quad N(\eta, z) = y_1' z_1 - F_{y_1} \eta_1 = 0.$$

It is easily shown that the expression $\eta_1 \bar{z}_1 - \bar{\eta}_1 z_1$ is a constant on $t_1 t_2$ for two pseudo-accessory extremaloids or for two accessory extremaloids. When this constant is zero the two sets will be called conjugate. The determinant of the quantities ξ_θ , $\eta_1(x_\theta^+)$, $z_1(x_\theta^+)$ or ξ_θ , $\eta_1(x_\theta^-)$, $z_1(x_\theta^-)$ in the equations (b') is a non-zero multiple of $\Omega_0(x_\theta)$. We see then that a pseudo-accessory extremaloid is uniquely determined by the initial values of η_1 , z_1 . A pseudo-accessory extremaloid becomes an accessory extremaloid when we add a suitable functional multiple of the set y_1' , F_{y_1} . Likewise an accessory extremaloid becomes a pseudo-accessory extremaloid if a suitable functional multiple of the set y_1' , F_{y_1} is added. It follows that two accessory extremaloids which coincide at $t = t_1$ differ by a functional multiple of the set y_1' , F_{y_1} . We note finally that if ρ is a function satisfying the continuity properties (a) and vanishing at corner values then the set $\rho y_1'$, ρF_{y_1} is an accessory extremaloid.

6. Fourth necessary conditions. In this section we shall discuss analogues of the Jacobi condition for the simplest problem of the calculus of variations. We shall deduce conditions analogous to those of Bliss, Hestenes, and Cope for the non-parametric problem as well as the conjugate point condition.

Let us consider as did Hestenes in the non-parametric case the problem with separated end-conditions. For this problem the second variation has the form

$$2\gamma_1[\eta(t_1)] - 2\gamma_2[\eta(t_2)] + \int_{t_1}^{t_2} 2\omega(\eta, \eta') dt.$$

The accessory end and transversality conditions become

$$(6:1) \quad \bar{\Psi}_\rho[\eta(t_1)] = 0 \quad (\rho = 1, \dots, \rho_0),$$

$$(6:2) \quad \bar{\Psi}_\sigma[\eta(t_2)] = 0 \quad (\sigma = \rho_0 + 1, \dots, \rho_0 + \sigma_0),$$

$$(6:3) \quad -\bar{\mathcal{L}}_1(t_1) d\eta_{11} + d\gamma_1 + \epsilon_\rho d\bar{\Psi}_\rho = 0,$$

$$(6:4) \quad \bar{\mathcal{L}}_1(t_2) d\eta_{12} - d\gamma_2 + \epsilon_\sigma d\bar{\Psi}_\sigma = 0.$$

Conditions (6:1) and (6:3) are equivalent to $n + \rho_0$ linear equations

$$(6:5) \quad -\bar{\mathcal{L}}_1(t_1) + \gamma_{1,11} + \epsilon_\rho \bar{\Psi}_{\rho,11} = 0,$$

$$\bar{\Psi}_\rho = 0.$$

If we adjoin the equation

$$\eta_1(t_1) F_{y_1}(t_1) - \bar{\mathcal{L}}_1(t_1) y_1'(t_1) = 0,$$

we obtain $n + \rho_0 + 1$ linear equations in $2n + \rho_0$ variables, $\eta_1(t_1), \bar{\mathcal{L}}_1(t_1), \epsilon_\rho$. If we assume the non-tangency condition, that is, that the matrix

$$||r_{\mu y_{11}} y_1'(t); r_{\mu y_{12}} y_1'(t_2)||$$

has rank two, then these equations are independent. Hence there are exactly $n - 1$ linearly independent solutions $\eta_{1k}(t_1)$, $\xi_{1k}(t_1)$, ϵ_{pk} , ($k = 1, \dots, n - 1$). It is readily verified that the sets $\eta_{1k}(t_1)$, $\xi_{1k}(t_1)$ are linearly independent. It follows by a slight extension of an argument made by Bliss (II, p. 84) that if E is non-singular these initial values determine a set of $n - 1$ linearly independent pseudo-accessory extremaloids conjugate in pairs. We may join the ends of these pseudo-accessory extremaloids with accessory extremaloids and obtain a linearly independent, mutually conjugate system η_{1k} , ξ_{1k} of accessory extremaloids satisfying the accessory end and transversality conditions at t_1 . This conjugate system is not unique, since a set $\rho y_1'$, ρF_{y_1} , $\rho(t_1) = \rho(x_0) = 0$ may be added to any accessory extremaloid of the system. Our conditions will be, however, independent of such possible changes. A similar argument provides us with a system $u_{1\ell}$, $v_{1\ell}$ ($\ell = 1, \dots, n - 1$) with the corresponding properties at t_2 . In terms of these systems we state the following necessary condition.

THEOREM 6.1. A non-singular extremaloid $E: y_1(t)$, $\lambda_0 = 1$, $\lambda_\beta(t)$ satisfying the non-tangency condition for a problem with separated end-conditions will be said to satisfy the condition IV₁ in case the inequality

$$(6:6) \quad (\xi_{1k} u_{1\ell} - \eta_{1k} v_{1\ell}) a_k b_\ell \geq 0 \quad (k, \ell = 1, \dots, n - 1)$$

holds for every set of values t , a_k , b_ℓ , with t on $t_1 t_2$ and

$$(6:7) \quad \eta_{1k}(t) a_k = u_{1\ell}(t) b_\ell + c y_1'(t^-) + d y_1'(t^+),$$

where $cd \geq 0$; furthermore, $cd = 0$ if t defines a corner where the bilinear form (6:6) vanishes. Every normal non-singular extremaloid associated with a minimizing arc for a problem with separated

end-conditions which satisfies the non-tangency condition must satisfy the condition IV₁.

If we make definitions analogous to the usual ones

$$\eta_1 = \eta_{1k} a_k, \quad u_1 = u_{1l} b_l, \quad \zeta_1 = \zeta_{1k} a_k, \quad v_1 = v_{1l} b_l,$$

and if further we define an admissible set of variations $\bar{\eta}_1$, satisfying the equations (5:1) by the equations

$$\begin{aligned} \bar{\eta}_1 &= \eta_1 + \rho y_1' \text{ on } t_1 t, \quad \rho(x_\theta) = 0 \quad (\theta = 0, \dots, r+1, x_\theta \neq t), \\ \bar{\eta}_1 &= u_1 + \rho y_1' \text{ on } t t_2, \quad \rho(t^-) + c = \rho(t^+) - d = 0, \end{aligned}$$

the usual manipulations suffice to show that

$$J_2(\eta) = (\zeta_{1k} u_{1l} - \eta_{1k} v_{1l}) a_k b_l + cd \Omega_0(t).$$

The truth of the theorem is then evident.

We shall say that the end-conditions for a non-singular extremaloid $E: y_1(t)$, $\lambda_0 = 1$, $\lambda_\rho(t)$ which satisfies the equations (1:2) and the condition I are conjugate in case there is an accessory extremaloid η_1, ζ_1 which satisfies the equations (5:1), (5:3) and for which $\eta_1 \neq \rho y_1'$ on $t_1 t_2$. We shall denote by IV_1' the condition IV_1 with the additional assumption that the end-conditions are not conjugate.

We come now to the notion of conjugate points. We shall say that a value t_3 defines a point y_3 conjugate to y_1 in case there is an accessory extremaloid η_1, ζ_1 and constants ϵ, δ, δ which satisfy the conditions

- (a) $\eta_1 \neq \rho y_1'$ on $t_1 t_3$,
- (b) $(\epsilon, \delta, \delta) \neq (0, 0, 0), \quad \delta \delta \geq 0$,
- (c) $\epsilon \eta_1(t_1) = 0, \epsilon \eta_1(t_3) = \delta y_1'(t_3^-) + \delta y_1'(t_3^+)$.

This definition is a slight departure from that given by Hefner (IV, p. 22) but this modification seems to be necessary if we

wish to consider abnormal arcs. By the usual manipulations and the use of Lemma 7.4 of Hefner (IV, p. 18) we can justify

THEOREM 6.2. If the arc of a non-singular extremaloid $E: y_1(t), \lambda_0 = 1, \lambda_p(t)$ is normal on every subinterval $t_1 t_3$ and minimizes the functional J then there is no point y_3 conjugate to y_1 between y_1 and y_2 .

An extremaloid $E: y_1(t), \lambda_0 = 1, \lambda_p(t)$ will be said to satisfy the condition IV_0 (IV_0') in case there is no point y_3 conjugate to y_1 between y_1 and y_2 (between y_1 and y_2 or at y_2).

We now introduce a condition which is analogous to one given by Bliss (II, p. 88). Let $\eta_{if}(t), \dot{\eta}_{if}(t)$ ($f = 1, \dots, h$) denote a maximal set of linearly independent accessory extremaloids which satisfy the equations (5:1) and which are linearly independent of the accessory extremaloids for which $\eta_1 \equiv \rho y_1'$ on $t_1 t_2$. We write $Q(z) = J_2(\eta_f z_f)$. We denote by IV_B the necessary condition $Q(z) \geq 0$. The symbol IV_B' will indicate that the quadratic form $Q(z)$ is positive definite.

It is also possible to give a formulation of a fourth necessary condition in terms of a boundary value problem. In order to do this we consider the problem of minimizing $J_2(\eta)$ in the class of admissible variations $\eta_1(t)$ which satisfy the equations (5:1), for which equations of the form $\eta_1' y_1' = \kappa_\theta$ hold on $x_\theta < t < x_{\theta+1}$ ($\theta = 0, \dots, r$), and are normed so that

$$\kappa_\theta \kappa_\theta + \int_{t_1}^{t_2} \eta_1 \eta_1 dt = 1 \quad (\theta = 0, \dots, r; i = 1, \dots, n).$$

The boundary value problem consisting of the Euler-Lagrange equations and transversality conditions for this problem may be shown to be of the type to which known theorems are applicable. This may be done, for example, by using a transformation employed by

Reid (IX, p. 78). Characteristic numbers for this problem will exist and have minimizing properties if the condition III' holds. In particular the least characteristic number for this problem is the minimum of $J_2(\eta)$, in the class described. We denote by IV_C (IV_C') the condition that this characteristic number be non-negative (positive). The usefulness of this condition will be more apparent when we have proved the following lemma.

LEMMA 6.1. Let $E: y_1(t)$, $\lambda_0 = 1$, $\lambda_\theta(t)$ be a non-singular extremaloid satisfying the equations (1:2), the non-tangency condition, and the conditions I and III'. Then the condition IV_C' is equivalent to the condition IV_*' .

To show that IV_C' implies IV_*' we note that corresponding to a non-tangential admissible variation η_1 satisfying the equations (5:1) we may choose a set $\bar{\eta}_1(t)$, κ_θ ($\theta = 0, \dots, r$) in the class described for which $\bar{\eta}_1 = \eta_1 + \int y_1'$ and $J_2(\bar{\eta}) = J_2(\eta)$. To show that IV_*' implies IV_C' we note first that because of the non-tangency condition a tangential variation satisfying the first two requirements of our class cannot satisfy the third. Thus the condition IV_*' says that $J_2(\eta) > 0$ for all variations η_1 in the class. The desired conclusion then follows from the minimizing property of the least characteristic number of our boundary value problem.

7. A theorem on accessory extremaloids. As an important step in the development of relations between the fourth necessary conditions we prove the following theorem.

THEOREM 7.1. If the extremaloid $E: y_1(t)$, $\lambda_0 = 1$, $\lambda_\theta(t)$ with $\Omega_0(x_\theta) < 0$ ($\theta = 1, \dots, r$) satisfies the conditions III' and IV_0' then there is a system of mutually conjugate accessory extremaloids $U_{1k}(t)$, $V_{1k}(t)$ ($k = 1, \dots, n - 1$) with the

determinant $|U_{1k} \ y_1'|$ neither vanishing nor changing sign on $t_1 t_2$.

An analogous theorem for the case of extremal arcs for the non-parametric problem has been proved by Reid (X, pp. 573-580). Our proof of this theorem is merely a slight extension of his argument. We begin by making definitions which provide an interesting method of determining conjugate points. We say that the order of abnormality on a subinterval $t't''$ of $t_1 t_2$ is the maximum number of linearly independent accessory extremaloids $u_{1\kappa}, v_{1\kappa}$ ($\kappa = 1, \dots, q$) with $u_{1\kappa} \equiv 0$ on $t't''$. It follows by the consideration of the equations (5:7) that the order of abnormality does not exceed m and that we have

$$(7:1) \quad \begin{aligned} v_{1\kappa}(T_1) \eta_1(T_1) &= v_{1\kappa}(T_2) \eta_1(T_2), \\ y_1'(t) v_{1\kappa}(t) &\equiv 0 \quad \text{on } t't'', \end{aligned}$$

where $\eta_1(t)$ is an arbitrary admissible variation and T_1 and T_2 lie on $t't''$. Define a non-increasing function $r(t)$ to be the order of abnormality on the subinterval $t_1 t$ ($t_1 < t \leq t_2$) of $t_1 t_2$. Clearly $r(t)$ has at most m point of discontinuity on $t_1 < t \leq t_2$. Denote these points by σ_q ($q = 1, \dots, g \leq m$), where $\sigma_q > \sigma_{q+1}$ ($q = 0, \dots, g$), where for convenience we have put $\sigma_0 = t_2$, $\sigma_{g+1} = t_1$. Finally let $r_q = r(\sigma_q)$ ($q = 0, \dots, g$). From the continuity of accessory extremaloids it follows that at a point of discontinuity $r(t)$ is continuous on the left.

Now it can be shown that there is a mutually conjugate family of accessory extremaloids u_{1k}, v_{1k} ($k = 1, \dots, n-1$) satisfying the conditions

$$\begin{aligned} u_{1k}(t_1) &= 0, \quad u_{1h} \equiv 0 \quad \text{on } t_1 \sigma_q \quad (h = 1, \dots, r_q; q = 0, \dots, g), \\ v_{1k}(t_1) v_{1l}(t_1) &= \delta_{kl} \quad (\delta_{kk} = 1, \delta_{kl} = 0 \text{ if } k \neq l) \\ &\quad (k, l = 1, \dots, n-1). \end{aligned}$$

We define a second family of mutually conjugate accessory extremaloids by the initial values,

$$u_{1k}^0(t_1) = v_{1k}(t_1), \quad v_{1k}^0(t_1) = y_1'(t_1)[v_{jk}(t_1)F_{y_j}(t_1)].$$

In terms of these we define families of mutually conjugate accessory extremaloids as follows

$$u_{1k}(t|q) = u_{1k}^0(t), \quad v_{1k}(t|q) = v_{1k}^0(t) \quad \text{for } k = 1, \dots, r_q, \\ u_{1k}(t|q) = u_{1k}(t), \quad v_{1k}(t|q) = v_{1k}(t) \quad \text{for } k = r_q + 1, \dots, n - 1.$$

We have then the following interesting method of determining conjugate points.

LEMMA 7.1. Let $E: y_1(t), \lambda_0 = 1, \lambda_\theta(t)$ be a non-singular extremaloid with $\Omega_0(x_\theta) < 0$ ($\theta = 1, \dots, r$). A value t_3 on the interval $\sigma_{q+1} < t_3 \leq \sigma_q$ ($q = 0, \dots, g$) defines a point y_3 conjugate to y_1 if and only if

$$|u_{1k}(t_3|q) \ y_1'(t_3^-)| \cdot |u_{1k}(t_3|q) \ y_1'(t_3^+)| \leq 0.$$

If the inequality holds there are constants $(c_k, e) \neq (0_k, 0)$, c, d with $cd \geq 0$ satisfying the equations

$$c_k u_{1k}(t_3|q) + c e y_1'(t_3^-) + d e y_1'(t_3^+) = 0.$$

Multiplying these equations by $v_{1f}(t_3)$ ($f = 1, \dots, r_q$), adding, and using equations (7:1) we obtain $c_f = 0_f$. Now it is readily verified that $\eta_1 = c_g u_{1g}$, $\zeta_1 = c_g v_{1g}$ ($g = r_q + 1, \dots, n - 1$), $\epsilon = 1$, $\delta = -ce$, $\delta = -de$ are effective in the definition of conjugate point or that t_3 defines a cusp. The converse is obvious if $\epsilon = 0$. For $\epsilon \neq 0$ it can be shown that we have $\epsilon \eta_1 = c_g u_{1g} + \rho y_1'$, $\epsilon \zeta_1 = c_g v_{1g} + \rho F_{y_1}$, and the conclusion follows readily.

The following lemma may be proved by slight extensions of arguments given by Bliss (I, p. 739).

LEMMA 7.2. THE TRANSFORMATION OF GLEBSCH. Let u_{1k} , v_{1k} ($k = 1, \dots, n - 1$) be mutually conjugate accessory extremaloids and $\eta_1(t)$ an arbitrary admissible variation for which there are functions $a_k(t)$, $\rho(t)$ of class C^1 between corner-values such that the equations

$$\eta_1 = u_{1k}a_k + \rho y_1'$$

hold on $t_1 t_2$. Then we have

$$\begin{aligned} \int_{t_1}^{t_2} 2\omega(\eta, \eta') dt &= \eta_1(v_{1k}a_k + \rho_{F_{y_1}})|_1^2 - \sum_{\theta=1}^r \rho_{\theta} \Omega_0(x_{\theta}) \\ &\quad + \int_{t_1}^{t_2} F_{r_1 r_j} v_1 v_j dt, \end{aligned}$$

where $v_i = a_k' u_{ik}$, and $\rho_{\theta} = \rho(x_{\theta}^-) \rho(x_{\theta}^+)$ (θ not summed).

These lemmas together with an examination of Reid's proof suffice to show that his method can be readily extended.

8. Conditions equivalent to the condition IV_{*}' . We shall show in this section that under suitable hypotheses the condition IV_{*}' is equivalent to IV_B' with IV_0' , to IV_1' , and to IV_C' .

When the non-tangency condition holds we see from Lemma 6.1 that the condition IV_C' holds if and only if the condition IV_{*}' holds. The condition IV_{*}' also implies IV_0' and IV_B' . The discussion made in Section 6 justifies this conclusion for the weak conditions. Neither the end-conditions nor the end-points are conjugate when IV_{*}' holds since a set defining such conjugacy would give the second variation the value zero, as may be seen by manipulations like those made in the proof of Theorem 6.1.

In order to show that the condition IV_B' with IV_0' implies the condition IV_{*}' for an extremaloid E satisfying III' and with

$\Omega_0(x_\theta) < 0$ ($\theta = 1, \dots, r$) we prove two lemmas which are analogous to certain ones used by Bliss (II, pp. 122-125).

LEMMA 8.1. If the end-points of the arc of a non-singular extremaloid E are not conjugate, then for every admissible variation $\eta_1(t)$ there is an accessory extremaloid $u_1(t), v_1(t)$ such that $\eta_1(t_s) = u_1(t_s)$ ($s = 1, 2$).

Let the order of abnormality on $t_1 t_2$ be q and $u_{1\kappa}, v_{1\kappa}$ ($\kappa = 1, \dots, q$) be linearly independent accessory extremaloids with $u_{1\kappa} \equiv 0$ on $t_1 t_2$. Choose $2n - 2 - q$ further accessory extremaloids $u_{1\sigma}, v_{1\sigma}$ ($\sigma = q + 1, \dots, 2n - 2$) so that the rank of the matrix

$$\begin{vmatrix} u_{1\kappa} & u_{1\sigma} & y_1' \\ v_{1\kappa} & v_{1\sigma} & F_{y_1} \end{vmatrix}$$

is $2n - 1$. Then when the end-points of the arc are not conjugate the rank of the matrix

$$\begin{vmatrix} u_{1\sigma}(t_1) & y_1'(t_1) & 0_1 \\ u_{1\sigma}(t_2) & 0_1 & y_1'(t_2) \end{vmatrix}$$

is $2n - q$. The columns of this matrix provide $2n - q$ linearly independent solutions of the q independent equations

$$v_{1\kappa}(t_2)\eta_1(t_2) - v_{1\kappa}(t_1)\eta_1(t_1) = 0.$$

Hence the end-values of η_1 are linear combinations of these solutions. A simple application of the final statement of Section 5 yields the desired set. This completes the proof of the lemma. A slight extension of this argument yields the more general

LEMMA 8.2. If the end-points of the arc of a non-singular extremaloid E are not conjugate, and if the functions $c_1(t)$ satisfy the relation

$$c_1(t_2) v_1(t_2) - c_1(t_1) v_1(t_1) = 0$$

for every accessory extremaloid u_1, v_1 for which $u_1 \equiv 0$ on $t_1 t_2$ then there is an accessory extremaloid U_1, V_1 such that $c_1(t_s) = U_1(t_s)$ ($s = 1, 2$).

We shall apply this lemma in the proof of a sufficiency theorem in Section 12.

We now prove the following lemma.

LEMMA 8.3. If E is an extremaloid with $\Omega_0(x_\theta) < 0$ ($\theta = 1, \dots, r$) which satisfies III' and has a family of mutually conjugate accessory extremaloids $U_{1k}(t), V_{1k}(t)$ ($k = 1, \dots, n-1$) with the determinant $|U_{1k} \ y_1'|$ neither vanishing nor changing sign on $t_1 t_2$, then an accessory extremaloid u_1, v_1 gives the integral

$$\int_{t_1}^{t_2} 2\omega(\eta, \eta') dt$$

a smaller value than any admissible variation $\eta_1(t)$ joining its ends and not the sum of u_1 and a tangential variation.

Expanding 2Ω by Taylor's formula and integrating by parts we find that

$$(8:1) \quad \int_{t_1}^{t_2} 2\omega(\eta, \eta') dt - \int_{t_1}^{t_2} 2\omega(u, u') dt = \int_{t_1}^{t_2} 2\omega(\eta - u, \eta' - u') dt.$$

Since the determinant mentioned in the lemma does not vanish nor change sign on $t_1 t_2$ there are functions $a_k(t), \rho(t)$ of class C^1 between corner values such that

$$(8:2) \quad \begin{aligned} \eta_1 - u_1 &= U_{1k} a_k + \rho y_1', \\ \rho(x_\theta^-) \rho(x_\theta^+) &\geq 0 \quad (\theta = 1, \dots, r \text{ not summed}). \end{aligned}$$

If we apply the transformation of Clebsch we find since III'

holds and $\Omega_0(x_\theta) < 0$ ($\theta = 1, \dots, r$) that the difference (8:1) is positive unless $\rho(x_\theta) = 0$, $U_{1k}a_k' = \sigma(t)y_1'$. These equations imply that the functions $a_k(t)$ are constants which vanish at t_1 and hence are identically zero. The lemma now follows from the equations (8:1) and (8:2).

With these lemmas and Theorem 7.1 the proof that IV_B' with IV_0' implies IV_*' is simple. First, by Theorem 7.1 there is a conjugate family of accessory extremaloids with the determinant mentioned in Lemma 8.3 neither vanishing nor changing sign on t_1t_2 . Now let $\eta_1(t)$ be a set of non-tangential admissible variations satisfying the equations (5:1). By Lemma 8.1 there is an accessory extremaloid u_1, v_1 such that $u_1(t_s) = \eta_1(t_s)$ ($s = 1, 2$). It is seen by Lemma 8.3 that we have $J_2(\eta) \geq J_2(u)$. Moreover, $J_2(u)$ is a value of the quadratic form $Q(z)$. Thus $J_2(\eta) > 0$ and the condition IV_B' with IV_0' implies the condition IV_*' .

If the end-conditions are separated and the non-tangency condition holds the arguments made in Section 6 together with the remarks at the beginning of this section suffice to show that the condition IV_*' implies the condition IV_1' . If also the condition III' holds and $\Omega_0(x_\theta) < 0$ ($\theta = 1, \dots, r$) then IV_1' implies IV_*' . The proof of this fact may be made by using Lemma 8.1, the transformation of Clebsch, and the following lemma.

LEMMA 8.4. If the end-conditions are separated and the extremaloid E satisfies the non-tangency condition, the conditions III' and IV_1' , and has $\Omega_0(x_\theta) < 0$ ($\theta = 1, \dots, r$), then there is a family of mutually conjugate accessory extremaloids U_{1k}, V_{1k} ($k = 1, \dots, n - 1$) with the determinant $|U_{1k} \ y_1'|$ neither vanishing nor changing sign on t_1t_2 and for which the inequalities

$$2\delta_1[U_{1k}(t_1)a_k] - U_{12}(t_1)a_2V_{1k}(t_1)a_k > 0,$$

$$-2\delta_2[U_{1k}(t_2)b_k] - U_{12}(t_2)b_2V_{1k}(t_2)b_k > 0$$

hold for all non-vanishing sets (a_k) , (b_k) which satisfy the equations

$$\bar{T}_\rho[U_{1k}(t_1)a_k] = 0, \quad \bar{T}_\sigma[U_{1k}(t_2)b_k] = 0.$$

To prove this we may follow Bliss and replace the accessory minimum problem by an equivalent problem which is normal by deleting a suitably selected set of the end-conditions (5:1) (II, p. 78). When this has been done the matrix of coefficients of the bilinear form (6:6) is non-singular. We may replace our conjugate systems by others for which this matrix is the identity matrix. Then the conjugate system given by the equations

$$U_{1k} = \eta_{1k} + u_{1k}, \quad V_{1k} = \zeta_{1k} + v_{1k}$$

has the properties described in the lemma as may be seen by slight modifications of a proof of an analogous theorem given by Bliss (II, pp. 112-116).

We summarize the results of this section in the following

THEOREM 8.1. If the extremaloid $E: y_1(t)$, $\lambda_0 = 1$, $\lambda_\theta(t)$ with $\Omega_0(x_\theta) < 0$ ($\theta = 1, \dots, r$) satisfies the condition III' then condition IV*' is equivalent to IV_B' with IV₀'. When also the non-tangency condition holds the conditions IV*' and IV_C' are equivalent. If further the end-conditions are separated then the conditions IV*' and IV₁' are equivalent.

9. A sufficiency theorem and a useful formula. We shall devote the remainder of this paper to a direct proof of the following

THEOREM 9.1. If the extremaloid $E: y_1(t), \lambda_0 = 1, \lambda_\beta(t)$ ($t_1 \leq t \leq t_2$) satisfies conditions I, II', III', IV*', the equations (1:3), the non-tangency condition, has $\Omega_0 < 0$ as its corner points $t = x_\theta$ ($\theta = 1, \dots, r$), and the arc $y_1 = y_1(t)$ does not intersect itself, then there is a neighborhood $\mathcal{T}$ of the arc $y_1 = y_1(t)$ in y -space and a neighborhood $\mathcal{N}$ in $2n$ -dimensional space of the ends of this arc such that the inequality $J(C) > J(E)$ holds for every admissible arc C in $\mathcal{T}$ with ends in $\mathcal{N}$ satisfying the equations (1:3) and different from E .

The method to be used is due to Hestenes (VII) who has recently given a corresponding proof for an extremal arc in the non-parametric problem of Bolza.¹

As a first step in the proof of this theorem we derive a formula relating the integral of J to the Hilbert invariant integral. To do this let us consider a one-parameter family of extremaloids

$$(9:1) \quad y_1 = y_1(t, b), \quad \lambda_0 = 1, \quad \lambda_\beta = \lambda_\beta(t, b)$$

whose end-values and corners all occur for the same values of the parameter t and are such that there is a neighborhood of the sets (t, b) satisfying $t_1 \leq t \leq t_2, b_1 \leq b \leq b_2$ in which, except at corners, the functions (9:1) and $y_{1t}(t, b)$ are of class C^1 and which have continuous derivatives with respect to b at values of t defining corners on the arcs (9:1). The integral of J evaluated on the family (9:1) is a differentiable function of b

$$I(b) = \int_{t_1}^{t_2} F[y(t, b), y'(t, b), \lambda(t, b)] dt$$

whose differential is given by the equation

¹The author has had the privilege of reading Hestenes' paper in manuscript. He wishes to express his appreciation of the simplicity which this work has made possible.

$$dI = F_{r_1} dy_1 \Big|_2^1.$$

It follows by integration that the Hilbert invariant integral,

$$I^*(C) = \int_C F_{r_1} dy_1,$$

satisfies the relation

$$(9:2) \quad I(E_1) - I(E_2) = I^*(C_1) - I^*(C_2),$$

where E_1 and E_2 are the extremaloids defined by b_1 and b_2 , respectively, and C_1 and C_2 are the loci of their end-points.

10. Mayer fields and a fundamental sufficiency theorem.

The definition of a field given here is similar to that used by Reid (IX). The fundamental sufficiency theorem follows Hestenes (VII).

A region $\mathcal{F}$ of y -space, together with a set of functions $p_1(y)$, $l_0 = 1$, $l_\theta(y)$, and r corner manifolds,

$$m_\theta: \quad y_1 = Y_{1\theta}(b) \quad (\theta = 1, \dots, r),$$

will be said to form a Mayer field in case the following properties are satisfied.

- (a) m_θ are non-singular, have no multiple points, do not intersect each other, and each m_θ divides $\mathcal{F}$ into two parts.
- (b) $p_1(y)$, $l_\theta(y)$ are of class C^1 between corner manifolds in $\mathcal{F}$ and approach finite limits on each side of each corner manifold.
- (c) The two limits p_1^- , p_1^+ of $p_1(y)$ at a point of a corner manifold are such that p_1^- and p_1^+ always determine directions on the same side of the corner manifold and never tangent to it.

- (d) The sets $[y, p(y)]$ for y in $\mathcal{F}$ are all admissible.
 (e) The functions $F_{r_1}[y, p(y), \mathbf{l}(y)]$ are continuous in $\mathcal{F}$.
 (f) The Hilbert integral

$$I^* = \int_{F_{r_1}} [y, p(y), \mathbf{l}(y)] dy_1$$

is independent of the path in $\mathcal{F}$.

Graves has noted that the condition (c) is equivalent to saying that the determinants

$$|Y_{1\theta a_k} \quad p_1^-|, \quad |Y_{1\theta a_k} \quad p_1^+|$$

are both of the same sign and neither zero along a corner manifold (III, p. 24). Between corner manifolds the solutions of the equations $dy/dt = p_1(y)$ are extremals and the condition (e) then implies that they may be pieced together to form extremaloids. These are called the extremaloids of the field.

In order to apply the methods of Hestenes it is convenient to adjoin to the equations (1:3) $2n - p$ further functions $\Psi_{p+h}(y_1, y_2)$ ($h = 1, \dots, 2n - p$) of class C''' in a neighborhood of the ends of E , vanishing there, and such that the matrix $\|\Psi_{\tau y_{1s}}\|$ ($\tau = 1, \dots, 2n; s = 1, 2$) is non-singular at the ends of E . The equations $\Psi_\mu = 0$, $\Psi_{p+h} = \alpha_h$ then have in a neighborhood of $(\alpha) = (0)$ unique solutions $y_{1s} = y_{1s}(\alpha)$. Furthermore, the equations

$$\Phi_\mu(\eta) = 0, \quad \Psi_{p+h}(\eta) = w_h$$

have unique solutions $\eta_1(t_s) = c_{1h}^s w_h$ with the matrix $\|c_{1h}^1; c_{1h}^2\|$ of rank $2n - p$. It is also clear that if the set $\eta_1(t)$ of admissible variations satisfy the equations (5:1), then

$$(10:1) \quad 2\delta[\eta(t_1), \eta(t_2)] = b_{gh} w_g w_h \quad (g, h = 1, \dots, 2n - p),$$

where b_{gh} are constants and $b_{gh} = b_{hg}$.

Let us now suppose that $\mathcal{T}$ is a neighborhood of E in y -space and A is a neighborhood of $(\alpha) = (0)$ in $(\alpha_1, \dots, \alpha_{2n-p})$ -space such that the points y_1, y_2 determined by (α) in A are in $\mathcal{T}$. Suppose further that for each element (α) in A the region $\mathcal{T}$ with the functions $p_1(y, \alpha)$, $l_0 = 1$, $l_\beta(y, \alpha)$ and the corner manifolds

$$m_\theta(\alpha): \quad y_1 = y_{1\theta}(y, \alpha) \quad (\theta = 1, \dots, r)$$

forms a Mayer field, and that E is an extremaloid of the field determined by $(\alpha) = (0)$. Define a function $W(\alpha)$ by the equation

$$W(\alpha) = g[y_1(\alpha), y_2(\alpha)] + \int_{C_{12}} Fr_1[y, p(y, \alpha), l(y, \alpha)] dy_1.$$

Using these definitions, the usual methods (II, pp. 107-108) suffice to justify the following theorem.

THEOREM 10.1. A FUNDAMENTAL SUFFICIENCY THEOREM. If for each (y) in $\mathcal{T}$ and each element (α) in A the inequality

$$(10;2) \quad \mathcal{E}[y, p(y, \alpha), l(y, \alpha); r] > 0$$

holds for every admissible set $(y, r) \neq (y, p^+)$, $(y, r) \neq (y, p^-)$, and the function $W(\alpha)$ has a proper relative minimum at $(\alpha) = (0)$, then the inequality $J(C) > J(E)$ holds for every admissible arc C in $\mathcal{T}$ not identical with E and having its end-values in a sufficiently small neighborhood $\mathcal{N}$ in $2n$ -dimensional space of those of E .

11. Construction of a family of Mayer fields. As a further step in the proof of Theorem 9.1 we devote this section to the proof of the following lemma.

LEMMA 11.1. Let $E: y_1(t)$, $\lambda_0 = 1$, $\lambda_\rho(t)$ ($t_1 \leq t \leq t_2$) be a non-singular extremaloid for which there exists a mutually conjugate set of accessory extremaloids $\eta_{ik}(t)$, $\zeta_{ik}(t)$ ($k = 1, \dots, n - 1$) with the determinant $|\eta_{ik} \ y_1'|$ neither vanishing nor changing sign on $t_1 t_2$. Let u_{ih} , v_{ih} ($h = 1, \dots, 2n - p$) be $2n - p$ further accessory extremaloids. Then there exists a $(3n - p - 1)$ -parameter family of extremaloids in canonical form

$$(11:1) \quad \begin{aligned} y_1 &= y_1(t, a, \alpha), & z_1 &= z_1(t, a, \alpha), \\ a &= (a_k), & \alpha &= (\alpha_h) \quad (k = 1, \dots, n - 1; h = 1, \dots, 2n - p), \end{aligned}$$

containing E for values $t_1 \leq t \leq t_2$, $a_k = 0_k$, $\alpha_h = 0_h$ and such that the functions (11:1) and y_{1t} , z_{1t} are of class C'' between corner manifolds

$$(11:2) \quad y_1 = y_1(x_\theta, a, \alpha) \quad (\theta = 1, \dots, r)$$

in a neighborhood of values (t, a, α) belonging to E . There are functions $\rho_k(t)$, $\rho_h^0(t)$ continuous on $t_1 t_2$, of class C' between and vanishing at corner values, such that the variations of the family (11:1) along E are given by the equations

$$(11:3) \quad \begin{aligned} y_{1a_k} &= \eta_{1k} + \rho_k y_1', & z_{1a_k} &= \zeta_{1k} + \rho_k^F y_1', \\ y_{1\alpha_h} &= u_{1h} + \rho_h^0 y_1', & z_{1\alpha_h} &= v_{1h} + \rho_h^{0F} y_1'. \end{aligned}$$

Moreover, there is a neighborhood $\mathcal{T}$ of E in y -space and a neighborhood A of $(\alpha) = (0)$ such that for every fixed set (α) in A the region $\mathcal{T}$ forms a Mayer field with the slope functions and multipliers

$$\begin{aligned} p_1(y, \alpha) &= y_{1t}[t(y, \alpha), a(y, \alpha), \alpha], \\ \ell_0 &= 1, \quad \ell_\rho(y, \alpha) = \lambda_\rho[t(y, \alpha), a(y, \alpha), \alpha], \end{aligned}$$

and the corner manifolds (11:2), where the functions $t(y, \alpha)$, $a_k(y, \alpha)$ are determined by the first n of the equations (11:1).

To prove this we note that it is clear from the discussion in Section 4 that there is a $(2n - 2)$ -parameter family of extremaloids

$$(11:5) \quad \begin{aligned} y_1 &= Y_1(t, b_1, \dots, b_n, c_1, \dots, c_n), \\ z_1 &= Z_1(t, b_1, \dots, b_n, c_1, \dots, c_n) \end{aligned}$$

containing E for values $b_1 = b_{10}$, $c_1 = c_{10}$, $t_1 \leq t \leq t_2$, the parameters (two of which are unessential) being the values of the functions Y_1, Z_1 at $t = t_1$, such that corners on the various extremaloids occur for the same values of t , and the functions (11:5) and Y_{1t}, Z_{1t} are of class C'' between corners in a neighborhood of values (t, b, c) defining E . The $(3n - p - 1)$ -parameter family (9:1) obtained from the family (11:5) by setting

$$\begin{aligned} b_1 &= b_{10} + \eta_{1k}(t_1)a_k + u_{1h}(t_1)\alpha_h, \\ c_1 &= c_{10} + \zeta_{1k}(t_1)a_k + v_{1h}(t_1)\alpha_h \end{aligned}$$

has the properties described in the lemma. For, by differentiating the identities,

$$b_1 = y_1(t_1, a, \alpha), \quad c_1 = z_1(t_1, a, \alpha)$$

with respect to a and α we find that the relations (11:3) hold at $t = t_1$ with $\rho_k(t_1) = 0$ and $\rho_h^0(t_1) = 0$. It follows that they hold on $t_1 t_2$ since the right and left members of (11:3) are accessory extremaloids. Thus the determinant $|y_{1a_k} \ y_{1t}|$ neither vanishes nor changes sign on $t_1 t_2$. The extension of the usual implicit function theorems described by Graves (III, pp. 25-27) insures us that the first n of the equations (11:1) have unique solutions $t(y, \alpha)$, $a_k(y, \alpha)$ which are of class C' between the

corner manifolds (11:2) for y in a neighborhood $\mathcal{F}_0$ of E and (α) in a neighborhood A_0 of $(\alpha) = (0)$. Denoting the multipliers of the family (11:2) by $\lambda_\theta(t, a, \alpha)$, a suitably chosen sub-neighborhood $\mathcal{F}$ of $\mathcal{F}_0$ with the corner manifolds (11:2) and the functions (11:4) forms a Mayer field for every (α) in a suitably chosen sub-neighborhood A of A_0 . The conditions (a)-(e) are readily verified for such a choice, and (f) follows from the result of Section 9 by the method used by Bliss (II, p. 106).

12. The minimum of the function $W(\alpha)$. The remainder of the proof of Theorem 9.1 is simple when we have established a proper relative minimum for the function $W(\alpha)$ at $(\alpha) = (0)$. We show in this section, by means of two lemmas, that under the hypotheses of Theorem 9.1 such is the case.

It is well known that the conditions $dW(0) = 0$, $d^2W(0) > 0$ for all $(d\alpha) \neq (0)$ are sufficient to insure a proper relative minimum of the function $W(\alpha)$ at $(\alpha) = (0)$. The first of our lemmas translates these conditions into the language of Section 11.

LEMMA 12.1. Let E be an extremaloid having the properties described in Lemma 11.1. The function $W(\alpha)$ will have a proper relative minimum at $(\alpha) = (0)$ if E satisfies the transversality condition (3:2) and the condition $B(w) > 0$ for every set $(w) \neq (0)$, where

$$B(w) = b_{gh}w_gw_h + [c_i(\xi_{ij}d_j + v_i) + (c_i - u_i)(v_i - \xi_{ij}e_j)]_1^2 \\ - \sum_{\theta=1}^r e_n(x_\theta^-)e_n(x_\theta^+) \Omega_0(x_\theta),$$

$$u_i = u_{ih}w_h, \quad v_i = v_{ih}w_h, \quad c_i = c_{ih}w_h, \quad d_i = d_{ih}w_h, \quad e_i = e_{ih}w_h \\ (i, j = 1, \dots, n; g, h = 1, \dots, 2n - p),$$

b_{gh} are constants defined by equations (10:1), and $c_{ih}(t)$, $d_{ih}(t)$,

$e_{ih}(t)$ are functions satisfying the conditions $c_{ih}(t_s) = c_{ih}^s$,

$$\eta_{ij}^{djh} = c_{ih} - u_{ih}, \quad \eta_{ij}^{ejh} = u_{ih}, \quad \zeta_{in} = \bar{y}_1, \quad \eta_{in} = y_1'.$$

Hestenes' proof of his Lemma 4.2 is easily extended to the present case.

Finally we again apply the methods of Hestenes to prove

LEMMA 12.2. If the condition IV_* ' holds for E and

$\Omega_0(x_\theta) < 0$ ($\theta = 1, \dots, r$), then there exists a mutually conjugate set of accessory extremaloids $\eta_{ik}(t)$, $\zeta_{ik}(t)$ ($k = 1, \dots, n - 1$) and a set of $2n - p$ accessory extremaloids $u_{ih}(t)$, $v_{ih}(t)$ ($h = 1, \dots, 2n - p$) such that the determinant $|\eta_{ik} \ y_1'|$ neither vanishes nor changes sign on $t_1 t_2$, and the quadratic form $B(w)$ described in Lemma 12.1 is positive definite.

By the arguments made in Sections 7 and 8 there is a set of mutually conjugate accessory extremaloids $\eta_{ik}(t)$, $\zeta_{ik}(t)$ such that the determinant mentioned in the lemma neither vanishes nor changes sign on $t_1 t_2$ when the condition IV_* ' holds. Let $U_{i\gamma}$, $V_{i\gamma}$ ($\gamma = 1, \dots, q$) be a maximal set of accessory extremaloids with $U_{i\gamma} \equiv 0$ on $t_1 t_2$ and with the matrix

$$\|V_{i\gamma}(t_2)c_{ih}^2 - V_{i\gamma}(t_1)c_{ih}^1\| \quad (\gamma = 1, \dots, q; h = 1, \dots, 2n - p)$$

of rank q . We can assume without loss of generality that

$$V_{i\gamma}(t_2)c_{ih}^2 - V_{i\gamma}(t_1)c_{ih}^1 = \delta_{\gamma h} \quad (\gamma = 1, \dots, q; h = 1, \dots, 2n - p).$$

Choose functions $c_{ih}(t)$ for which $c_{ih}(t_s) = c_{ih}^s$ ($s = 1, 2$).

Since the end-points are not conjugate when the condition IV_* '

holds the hypotheses of Lemma 8.2 are satisfied for $c_{i\sigma}(t)$

($q < \sigma \leq 2n - p$). Thus there are accessory extremaloids $u_{i\sigma}$,

$v_{i\sigma}$ ($\sigma = q + 1, \dots, 2n - p$) such that $c_{i\sigma}(t_s) = u_{i\sigma}(t_s)$.

Then the $2n - p$ accessory extremaloids $u_{1\gamma} = \epsilon U_{1\gamma} \equiv 0$, $v_{1\gamma} = \epsilon V_{1\gamma}$; $u_{1\sigma}$, $v_{1\sigma}$, with η_{1k} , ζ_{1k} make the quadratic form $B(w)$ positive definite for ϵ sufficiently large and positive. For since $u_{1\gamma} \equiv 0$, we have $e_{1\gamma} \equiv 0$, $d_{k\sigma} \equiv 0$, and $d_{n\sigma}(t_s) = 0_{s\sigma}$. We note first that the final term in $B(w)$ is non-negative since the determinant mentioned in the lemma does not change sign and $\Omega_0(x_0) < 0$ ($\theta = 1, \dots, r$). If we make the definitions

$$\begin{aligned} A_{\gamma\alpha} &= b_{\gamma\alpha} + \frac{1}{2}[c_{1\gamma}\zeta_{1j}d_{j\alpha} + c_{1\alpha}\zeta_{1j}d_{j\gamma}]_1^2 \quad (\gamma, \alpha = 1, \dots, q), \\ 2B_{\tau\gamma} &= 2b_{\tau\gamma} + [c_{1\tau}\zeta_{1j}d_{j\gamma} - c_{1\gamma}(\zeta_{1j}e_{j\tau} - 2v_{1\tau})]_1^2, \\ C_{\sigma\tau} &= b_{\sigma\tau} + \frac{1}{2}[u_{1\sigma}v_{1\tau} + u_{1\tau}v_{1\sigma}]_1^2 \quad (\sigma, \tau = q+1, \dots, 2n-p) \end{aligned}$$

it is easily seen that

$$B(w) \geq (A_{\gamma\alpha} + 2\epsilon\delta_{\gamma\alpha})w_{\gamma}w_{\alpha} + 2B_{\tau\gamma}w_{\gamma}w_{\tau} + C_{\sigma\tau}w_{\sigma}w_{\tau}.$$

Here the values of $A_{\gamma\alpha} = A_{\alpha\gamma}$, $B_{\tau\gamma} = B_{\gamma\tau}$, $C_{\sigma\tau} = C_{\tau\sigma}$ are independent of ϵ . If we set $\eta_1 = u_{1\sigma}w_{\sigma}$, $\zeta_1 = v_{1\sigma}w_{\sigma}$, then η_1 satisfy the equations (5:1) and we have

$$C_{\sigma\tau}w_{\sigma}w_{\tau} = \eta_1\zeta_1|_1^2 = J_2(\eta) > 0,$$

by condition IV*' unless η_1 is a tangential variation. In this case the non-tangency condition implies that $w_{\sigma} = 0$. Hence $C_{\sigma\tau}w_{\sigma}w_{\tau}$ is positive definite. The theory of quadratic forms then assures us that the form $B(w)$ will be positive definite for ϵ sufficiently large and positive.

The proof of Theorem 9.1 is now easy. By Lemmas 11.1, 12.1, and 12.2 there are neighborhoods $\mathcal{F}$ of E and A of $(\alpha) = (0)$ and a set of slope functions and multipliers (11:4) forming a Mayer field for every (α) in A and such that $W(\alpha)$ has a proper

minimum at $(\alpha) = (0)$. The arc E is an extremaloid of the field determined by $(\alpha) = (0)$. Moreover, by taking δ and A sufficiently small we can be assured that the inequality (10:2) holds, as may be seen by a slight extension of an argument of Graves (III, p. 8). Thus Theorem 10.1 implies Theorem 9.1.

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